SECONDARY PHYSICS

TESLA SECONDARY SIMULATION PROJECT

Document pn1401, version 0.1h. *DRAFT*. See [1] for the latest revision of this document.

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1. Introduction

In this document, which follows on from pn2511, we explore the theory of resonating solenoids in a little more detail. We begin by setting up the integral equation for the lossy resonator, which leads to an elegant description of the system in terms of integral operators. The analysis is extended to include a coupled primary resonator, as well as various types of external feed. We use the integral formulation to examine the normal mode spectrum of the Tesla coil and demonstrate how the time domain behaviour can be determined.

Readers should have read pn2511, and will need some familiarity with the use of linear operators on complex vector spaces.

2. The Integral Equation

We will begin by setting out the full differential equations for the secondary resonator, which were introduced in pn2511, but we now include additional terms for a series loss resistance, and for a number of feed arrangements. For a coil of length h, we have

$$\begin{aligned}\underbrace{\frac{\partial}{\partial r}I(r,t)}_{\text{Current gradient}} &= -\underbrace{C_{ext}(r)\frac{\partial}{\partial t}V(r,t)}_{\text{external capacitance current}} - \underbrace{C_{tor}(r)\frac{\partial}{\partial t}\left(V(r,t)-V(h,t)\right)}_{\text{toroid capacitance current}} \\ &\quad - \underbrace{\int_0^h C_{int}(r,s)\frac{\partial}{\partial t}(V(r,t)-V(s,t))\,ds}_{\text{internal capacitance current}}\end{aligned} \tag{2.1}$$

$$\begin{aligned}\underbrace{\frac{\partial}{\partial a}V(a,t)}_{\text{Voltage gradient}} &= -\underbrace{R_s(a)I(a,t)}_{\text{Ohm's law}} - \underbrace{\int_0^h M(a,b)\frac{\partial}{\partial t}I(b,t)\,db}_{\text{self induced EMF}} \\ &\quad - \underbrace{M_p(a)\frac{d}{dt}I_{pri}(t)}_{\text{primary induced EMF}} - \underbrace{\int_0^a \delta(b-n)\,db\,V_s(t)}_{\text{series feed voltage}}\end{aligned} \tag{2.2}$$

Date: 14th January 2001.

Produced by Paul Nicholson as part of the Tesla Secondary Simulation Project, http://www.abelian.demon.co.uk/tssp/.

where $C_{int}(x,y)$, $C_{tor}(x)$, $M(x,y)$, $M_p(x)$, and $C_{ext}(x,y)$ are the density functions describing the distributed physical reactances of the resonator, as defined in pn2511, and $R_s(x)$ is the resistance per unit length at the point x on the solenoid. In these equations, and those to follow, x, y, r, s, a, and b, are position variables which range 0 to h. The first equation applies Kirchhoff's law to every point on the coil to obtain an expression of conservation of charge, and the second equation applies Ohm's law and the induction law to every point of the coil to obtain the voltage gradient. We include terms in the voltage gradient to represent induction due to a current $I_p(t)$ in a coupled primary winding, acting through its distributed mutual inductance function $M_p(a)$, and we allow for a general series feed arrangment by integrating a Dirac delta function placed at the feedpoint n of a series voltage $V_s(t)$, If we integrate the current differential 2.1 along the coil, from the base at $r = 0$ up to a point $r = x$, we get,

$$\begin{aligned} I(x,t) =& I(0,t) + \int_0^x \frac{\partial}{\partial r} I(r,t)\, dr \\ =& I_{base}(t) - \int_0^x C_{ext}(r) \frac{\partial}{\partial t} V(r,t)\, dr \\ & - \int_0^x C_{tor}(r) \frac{\partial}{\partial t} \left(V(r,t) - V(h,t) \right)\, dr \\ & - \int_0^x \int_0^h C_{int}(r,s) \frac{\partial}{\partial t} (V(r,t) - V(s,t))\, ds\, dr \end{aligned} \tag{2.3}$$

where we use $I_{base}(t)$ as an alias for the base current $I(0,t)$. This equation can be re-arranged to give

$$\begin{aligned} I(x,t) =& I_{base}(t) \\ & - \int_0^h \Bigg\{ H(x-r) \left(C_{ext}(r) + C_{tor}(r) + \int_0^h C_{int}(r,s)\, ds \right) \\ & + \int_0^h C_{tor}(b)\, db\, \delta(r-h) + \int_0^x C_{int}(s,r)\, ds \Bigg\} \frac{\partial}{\partial t} V(r,t)\, dr \end{aligned} \tag{2.4}$$

in which H is the Heaviside function employed to extend the integration limits, ie

$$\begin{aligned} H(a) =& 1 \quad \text{if} \quad a > 0 \\ & 0 \quad \text{if} \quad a <= 0 \end{aligned} \tag{2.5}$$

Now if we apply Kirchhoff's law to the coil base, we find I_{base} is given by

$$I_{base}(t) = \underbrace{\int_0^h C_{ext}(r) \frac{\partial}{\partial t} V(r,t)\, dr}_{\text{total external displacement}} + \underbrace{\left(G_l + C_{top} \frac{\partial}{\partial t} \right) V(h,t)}_{\text{load current}} - \underbrace{I_{tf}(t)}_{\text{top feed}} \tag{2.6}$$

where $I_{tf}(t)$ is an externally applied top feed current, if any. Using this expression to eliminate I_{base} in 2.4, and rearranging a little, gives us

$$I(x,t) = \int_0^h C(x,r) \frac{\partial}{\partial t} V(r,t)\, dr + \int_0^h \delta(r-h) G_l V(r,t)\, dr - I_{tf}(t) \tag{2.7}$$

where the kernel function $C(x,r)$ of the first integral is the capacitance density,

$$\begin{aligned}
C(x,r) =& H(r-x)C_{ext}(r) \\
&+ \delta(r-h)\left\{C_{top} + \int_0^x C_{tor}(b)\,db\right\} \\
&- H(x-r)\left\{C_{tor}(r) + \int_0^h C_{int}(r,s)\,ds\right\} \\
&+ \int_0^x C_{int}(s,r)\,ds
\end{aligned} \tag{2.8}$$

We can simplify the notation considerably if we regard the current and voltage functions $I(x,t)$ and $V(x,t)$ as vectors in an infinite dimensional vector space of functions. Adopting a convention of writing operators with a hat symbol, we can define two operators, $\hat{C}$ and $\hat{G}_l$, such that

$$\begin{aligned}
\hat{C}f &\equiv \int_0^h C(x,r)f(r,t)\,dr \\
\hat{G}_l f &\equiv \int_0^h \delta(r-h)G_l f(r,t)\,dr
\end{aligned} \tag{2.9}$$

each of which operates on a voltage profile vector f. With the operator $\hat{C}$, we have a mathematical object which encapsulates the entire capacitive reactance of the resonator, both the distributed and lumped parts together, so we'll call $\hat{C}$ the self-capacitance operator. $\hat{G}_l$ is an operator which describes the coupling of the top-end load conductance to the resonator. If for some reason shunt conductance is distributed along a particular coil, then the delta function in the kernel of $\hat{G}_l$ can be replaced by the appropriate distribution function. We can use these operators to rewrite 2.7 as

$$I(x,t) = \left(\frac{\partial}{\partial t}\hat{C} + \hat{G}_l\right)V - I_{tf}(t) \tag{2.10}$$

which shows how the two operators together transform a voltage distribution into a current distribution. We will return to this equation shortly, but first we apply a similar procedure to the voltage gradient of equation 2.2. Starting with integration from the base up to a point r, we get

$$\begin{aligned}
V(r,t) =& V(0,t) + \int_0^r \frac{\partial}{\partial a}V(a,t)\,da \\
=& V_{base}(t) - R_b I(0,t) \\
&- \int_0^r R_s(a)I(a,t)\,da \\
&- \int_0^r \int_0^h M(a,b)\frac{\partial}{\partial t}I(b,t)\,db\,da \\
&- \int_0^r M_p(a)\,da\,\frac{d}{dt}I_{pri}(t) \\
&- \int_0^r \int_0^a \delta(b-n)\,db\,da\,V_s(t)
\end{aligned} \tag{2.11}$$

where R_b is the effective series resistance of the ground and $V_{base}(t)$ is an arbitrary base drive voltage. This integral, after a bit of manipulation, becomes

$$
\begin{aligned}
V(r,t) =& V_{base}(t) - \int_0^h \left(\delta(a)R_b + H(r-a)R_s(a)\right) I(a,t)\,da \\
& - \int_0^h \int_0^r M(b,a)\,db\,\frac{\partial}{\partial t} I(a,t)\,da \\
& - \int_0^r M_p(a)\,da\,\frac{d}{dt} I_{pri}(t) \\
& - \int_0^r \int_0^a \delta(b-n)\,db\,da\,V_s(t)
\end{aligned}
\tag{2.12}
$$

Just as we did for the current integral, we define two integral operators, which are the duals of those in 2.9, as follows:

$$
\begin{aligned}
\hat{L}f &\equiv - \int_0^h \int_0^r M(b,a)\,db\; f(a,t)\,da \\
\hat{R}f &\equiv - \int_0^h \left(\delta(a)R_b + H(r-a)R_s(a)\right) f(a,t)\,da
\end{aligned}
\tag{2.13}
$$

which gives us a self-inductance operator $\hat{L}$, and the operator $\hat{R}$ which describes how the resonator couples to the series loss resistances - the distributed $R_s(a)$ and the lumped ground circuit resistance R_b. Just as with the dual operator $\hat{G}_l$, the delta function in $\hat{R}$ can be replaced to accomodate any arbitrary extra series resistances.

Using these two operators, we can write 2.12 as

$$
\begin{aligned}
V(r,t) =& \left(\frac{\partial}{\partial t}\hat{L} + \hat{R}\right) I \\
& + V_{base}(t) \\
& - \int_0^r M_p(a)\,da\,\frac{d}{dt} I_{pri}(t) \\
& - \int_0^r \int_0^a \delta(b-n)\,db\,da\,V_s(t)
\end{aligned}
\tag{2.14}
$$

We can now combine the integral expressions for current and voltage, to obtain a single equation for the resonator. We choose to eliminate the voltages to get an equation in the currents, but the following procedure can just as easily be applied the other way round.

Substituting 2.14 into 2.10 to eliminate the coil voltages gives

$$
\begin{aligned}
I(x,t) =& \left(\frac{\partial}{\partial t}\hat{C} + \hat{G}_l\right)\left(\frac{\partial}{\partial t}\hat{L} + \hat{R}\right) I \\
& + \left(\frac{\partial}{\partial t}\hat{C} + \hat{G}_l\right) \hat{1} V_{base}(t) \\
& - \left(\frac{\partial}{\partial t}\hat{C} + \hat{G}_l\right) \int_0^r M_p(a)\,da\,\frac{d}{dt} I_{pri}(t) \\
& - \left(\frac{\partial}{\partial t}\hat{C} + \hat{G}_l\right) \int_0^r \int_0^a \delta(b-n)\,db\,da\,V_s(t) \\
& - I_{tf}(t)
\end{aligned}
\tag{2.15}
$$

We can simplify this equation further by defining a uniform set of feed coupling operators, as follows,

$$
\begin{aligned}
\hat{v_b} f &\equiv \left(\frac{\partial}{\partial t}\hat{C} + \hat{G}_l\right)\hat{1} f(t) \quad \Big\} \text{ Base feed} \\
\hat{v_p} f &\equiv -\left(\frac{\partial}{\partial t}\hat{C} + \hat{G}_l\right)\int_0^r M_p(a)\, da\, \frac{d}{dt} f(t) \quad \Big\} \text{ Primary feed} \\
\hat{v_s} f &\equiv -\left(\frac{\partial}{\partial t}\hat{C} + \hat{G}_l\right)\int_0^r \int_0^a \delta(b-n)\, db\, da\, f(t) \quad \Big\} \text{ Series feed} \\
\hat{v_t} f &\equiv -\,\hat{1} f(t) \quad \Big\} \text{ Top feed}
\end{aligned}
\tag{2.16}
$$

Each of these operators converts an applied drive current or voltage, as appropriate, into a resonator current distribution, and in fact gives the current distribution in the coil due to the drive signal alone, ie the coil current with the current due to the coil's reaction to it removed. We can summarise all possible feed arrangements with the definition of a general feed, or forcing function

$$
v(x,t) = \hat{v_b} I_{base}(t) + \hat{v_p} I_{pri}(t) + \hat{v_s} V_s(t) + \hat{v_t} I_{tf}(t)
\tag{2.17}
$$

so that, when we wish to talk about an arbitrary feed, we can just use $v(x,t)$ and the reader can take that to be any desired mix of the available options in 2.17. For example, for a center-fed bipolar coil, we would use the forcing function

$$
v(x,t) = \hat{v_s} V_s(t) = \left(\frac{\partial}{\partial t}\hat{C} + \hat{G}_l\right)\int_0^r \int_0^a \delta(b-h/2)\, db\, da\, V_s(t)
\tag{2.18}
$$

in which the delta function $\delta(b-h/2)$ inserts the drive signal $V_s(t)$ at the center of the coil. We will often abbreviate $v(x,t)$ as a vector v, and doing so, we can now write 2.15 as

$$
I = v + \left(\frac{\partial}{\partial t}\hat{C} + \hat{G}_l\right)\left(\frac{\partial}{\partial t}\hat{L} + \hat{R}\right) I
\tag{2.19}
$$

and if we define an operator $\hat{A}$, such that

$$
\begin{aligned}
\hat{A} f &\equiv \left(\frac{\partial}{\partial t}\hat{C} + \hat{G}_l\right)\left(\frac{\partial}{\partial t}\hat{L} + \hat{R}\right) f \\
&\equiv \left(\frac{\partial^2}{\partial t^2}\hat{C}\hat{L} + \frac{\partial}{\partial t}\left\{\hat{C}\hat{R} + \hat{G}_l\hat{L}\right\} + \hat{G}_l\hat{R}\right) f
\end{aligned}
\tag{2.20}
$$

we obtain a fundamental equation for the solenoid,

$$
I = v + \hat{A} I
\tag{2.21}
$$

Any current function $I(x,t)$ supportable by the solenoid must be a solution of this equation. Note that $\hat{A}$ and v are given by a particular resonator and drive function, respectively. This equation provides a very concise description of the system. It simply says that the resonator current distribution I is the sum of the current distribution v due to the drive alone, plus the current $\hat{A}I$ induced through the reaction of the coil to its own current. The operator $\hat{A}$ completely describes the resonator, including its damping and the effects of termination load, so it's reasonable for us to refer to it as the *solenoid operator*. We'll take a closer look at this important operator in the next section.

3. The solenoid operator

Each of the terms in $\hat{A}$ is a spatial integral over the length of the coil, and in fact equation 2.21 is an example of a Fredholm integral equation of the second kind. We can write $\hat{A}$ explicitly as an integral, with

$$\begin{aligned} \hat{A}f &\equiv \left(\frac{\partial^2}{\partial t^2}\hat{C}\hat{L} + \frac{\partial}{\partial t}\left\{\hat{C}\hat{R} + \hat{G}_l\hat{L}\right\} + \hat{G}_l\hat{R}\right) f \\ &\equiv \int_0^h g(x,a) f(a)\, da \end{aligned} \tag{3.1}$$

where the kernel function $g(x,a)$ is the time dependent Green's function

$$g(x,a) \equiv F(x,a)\frac{\partial^2}{\partial t^2} - K(x,a)\frac{\partial}{\partial t} - G_l R_s(a) \tag{3.2}$$

in which the two real scalar functions of position, F and K, are given by

$$\begin{aligned} F(x,a) = &\int_x^h \int_0^h C_{int}(r,s) \int_r^s M(b,a)\, db\, ds\, dr \\ &- \int_x^h (C_{ext}(r) + C_{tor}(r)) \int_0^r M(a,b)\, db\, dr \\ &- \int_0^h C_{tor}(r) \int_r^h M(a,b)\, db\, dr \\ &+ \left(\int_x^h C_{tor}(r)\, dr - C_{top}\right) \int_0^h M(a,b)\, db \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} K(x,a) = &R_s(a) \int_x^h (C_{ext}(r) + C_{tor}(r))\, H(r-a)\, dr \\ &- R_s(a) \int_x^h \int_0^h C_{int}(r,s)\xi(r,s,a)\, ds\, dr \\ &+ \int_0^h C_{tor}(r) R_s(a) H(a-r)\, dr \\ &+ R_s(a) C_{top} - R_s(a) \int_x^h C_{tor}(r)\, dr \\ &+ G_l \int_0^h M(a,b)\, db \end{aligned} \tag{3.4}$$

The function $F(x,y)$, which is the kernel of the operator $\hat{C}\hat{L}$, is clearly a physical constant for any given resonator and serves to describe the reactive coupling of the coil to itself. $F(x,y)$ is independent of the electrical termination arrangements and is independent of the resonator's loss. The damping of the coil is given by $K(x,y)$, which describes how the loss is distributed along the coil. The term in K with coefficient G_l describes how the damping due to the load resistance is distributed along the coil through the mutual inductance M, and because of this term, K is not a constant for the resonator but varies with the termination conditions. K is the kernel of the damping operator $\hat{C}\hat{R} + \hat{G}_l\hat{L}$.

4. Solutions

We wish to determine I in 2.21, which is the response of the resonator to the given drive function v. It is immediately clear that any such solution is not unique. Suppose for example that a current distribution I_f is a solution to 2.21, ie

$$I_f = v + \hat{A} I_f \tag{4.1}$$

then we can generate other solutions - infinitely many - by adding to I_f a current distribution ϕ which is an eigenfunction of the solenoid operator $\hat{A}$. In fact we can add to I_f any linear combination of the eigenfunctions of $\hat{A}$ and still obtain a current profile which satisfies the integral equation, ie an allowable current profile. An eigenfunction ϕ of $\hat{A}$ is a vector which satisfies the homogeneous equation

$$\phi = \hat{A}\phi \tag{4.2}$$

which is the solenoid equation obtained when the drive is removed, ie v is set to zero, so that the equation describes a free resonance of the solenoid. The current profile vectors ϕ which satisfy this equation therefore correspond to the free resonances, or normal modes, of the solenoid. The homogeneous equation therefore implies that the current in an undriven resonator is strictly confined to being a linear combination of these eigenfunctions - in other words a superposition of normal modes. If we have a solution I_f to the current profile of the driven equation, then we can add ϕ to it to form another solution,

$$v + \hat{A}(I_f + \phi) = v + \hat{A} I_f + \hat{A}\phi = I_f + \hat{A}\phi = I_f + \phi \tag{4.3}$$

and it is clear that any linear combination of different ϕ will also work in the above equation.

Therefore, given a particular solution I_f of the forced response, the general solution to the solenoid equation is

$$I = I_f + \sum_{n=1}^{\infty} \lambda_n \phi_n \tag{4.4}$$

where the λ_n are amplitude coefficients which must be chosen to match the initial conditions. We now look at how to find the particular forced response solution I_f, which depends on the driving function v.

We can rewrite the integral equation as

$$(\hat{1} - \hat{A}) I = v \tag{4.5}$$

where $\hat{1}$ is the identity operator in this vector space, which can be written in our integral representation as

$$\hat{1} f(x,t) \equiv \int_0^h \delta(x-a) f(a,t)\, da \tag{4.6}$$

Now the solution to 4.5 is obtained formally by applying an inverse operator to both sides, as in

$$I = (\hat{1} - \hat{A})^{-1} v \tag{4.7}$$

All we need to do is to calculate this inverse operator, which is given by the series expansion

$$(\hat{1} - \hat{A})^{-1} = \hat{1} + \hat{A} + \hat{A}^2 + \hat{A}^3 + \dots \tag{4.8}$$

so that our solution is

$$I = v + \hat{A}v + \hat{A}^2 v + \hat{A}^3 v + \cdots = \sum_0^\infty \hat{A}^n v \tag{4.9}$$

Writing this out explicitly in the integral formulation gives

$$\begin{aligned} I(x,t) = v(x,t) &+ \int_0^h g(x,a_1) v(a_1,t)\, da_1 \\ &+ \int_0^h g(x,a_2) \int_0^h g(a_2,a_1) v(a_1,t)\, da_1\, da_2 \\ &+ \int_0^h g(x,a_3) \int_0^h g(a_3,a_2) \int_0^h g(a_2,a_1) v(a_1,t)\, da_1\, da_2\, da_3 + \ldots \end{aligned} \tag{4.10}$$

This suggests an iterative formulation for I, by setting

$$\begin{aligned} v_0 &= v \\ v_n &= \hat{A} v_{n-1} \end{aligned} \tag{4.11}$$

so that

$$I = v_0 + v_1 + v_2 + \ldots \tag{4.12}$$

When v is well away from any eigenfunction, this series converges rapidly, requiring only 2 or 3 terms to achieve good accuracy. In the vicinity of an eigenfunction, this convergence is very slow, and problems with accumulation of numerical precision errors occur before a satisfactory solution is reached. For practical computations a direct solution of 4.5 by Gaussian elimination is fast and reliable, but the so-called *Neumann series* 4.11 serves well to illustrate how the forcing function drives the resonance.

If we put together the forced response and free resonant solutions, we end up with a general solution for the solenoid current, under any conditions, which has the form

$$I = \underbrace{\sum_{n=0}^{\infty} \hat{A}^n v}_{\text{forced response}} + \underbrace{\sum_{n=0}^{\infty} \lambda_n \phi_n}_{\text{free resonance}} \tag{4.13}$$

so that the response of the coil, ie the current vector I, is the combination of the forced response of the coil obtained by the action of the solenoid operator on the drive current v, plus an arbitrary amount of free resonance, set by the amplitude coefficients λ_n, which are themselves chosen to match the initial conditions. Due to the linearity of $\hat{A}$, the forced response can contain only those frequency components which are present in the drive function v. The only other frequencies which can appear in I are the free-resonant frequencies, ie those of the eigenfunctions ϕ_n of $\hat{A}$. For any real solenoid, its operator $\hat{A}$ contains a damping term (the coefficient of $\partial/\partial t$), and as a result, the free-resonance components of I will suffer an exponential decay of their amplitudes. Therefore, regardless of the initial conditions, after a sufficient time has elapsed, only the forced response component of I will remain. Figure 5.1 shows a typical response of a resonator to a sinusoidal driving function which begins at $t = 0$. The initial transient response excites a free resonance, and because the drive frequency is chosen so that it does not coincide with a free

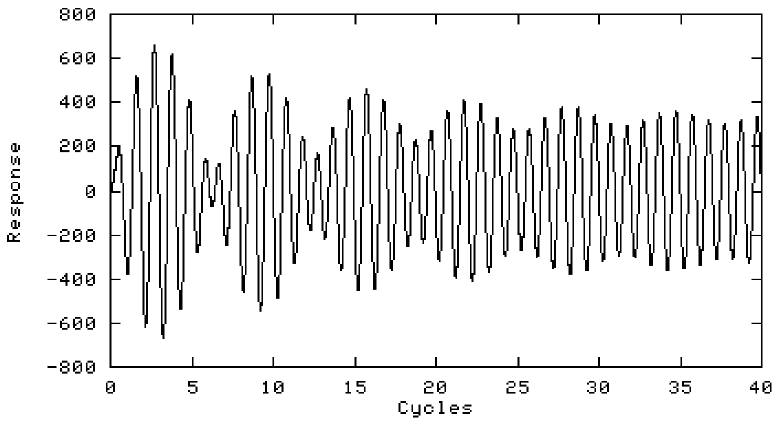

FIGURE 5.1. Forced response in the time domain

resonance, we see a beat envelope appear. After a number of cycles, the free resonance has decayed ($Q = 36$) to leave only the forced response of the coil.

5. Normal Modes

A basic characteristic of the normal modes of all linear resonating systems is that all parts of the resonator move with the same angular frequency, although they may have different phases. Mathematically, this means that the eigenfunctions are separable into a product of a spatial distribution function and a time dependent function. Furthermore, in view of the damping term - the function $K(x, y)$ embedded in the solenoid operator, we expect the time dependent part of the eigenfunction to contain an exponential decay term. Therefore we can expect the general form of the eigenfunctions to be

$$I_n(x,t) = \Re\left\{\bar{I}_n(x)e^{j\omega_n t}e^{-\alpha_n t}\right\} \tag{5.1}$$

where ω_n is the angular frequency and α_n the decay rate of the nth eigenfunction, which has a spatial amplitude and phase distribution given by the complex function $\bar{I}_n(x)$. We will combine the exponents into a single complex frequency variable,

$$\gamma_n = j\omega_n - \alpha_n \tag{5.2}$$

Combining the angular frequency and the decay rate into a single complex quantity like this allows us to treat both harmonic and exponential behaviour on an equal footing - the utility of which will become apparent, and the reader may notice the similarity to the variable of the Laplace transform. The eigenfunctions can also be written as

$$I_n(x,t) = \frac{1}{2}\left\{\bar{I}_n(x)e^{\gamma_n t} + \bar{I}_n^*(x)e^{\gamma_n^* t}\right\} \tag{5.3}$$

where the addition of the conjugate cancels out the imaginary component, thus achieving the desired projection to the real axis. The conjugate second term provides the negative frequency contributions, since

$$\gamma^* = -j\omega - \alpha \tag{5.4}$$

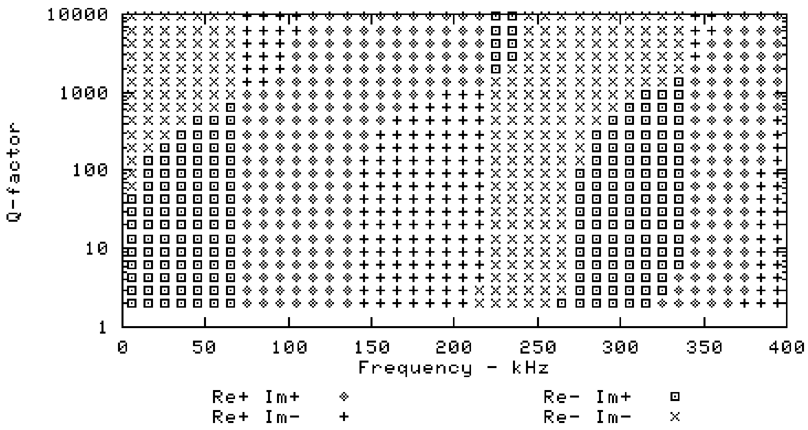

FIGURE 5.2. Landscape of det(A-1)

This conjugate or negative frequency term can also be thought of as describing a wave travelling in the opposite direction along the solonoid to that represented by the positive frequency term, with the two travelling waves being of equal amplitude and conjugate phase, thus forming the standing wave expected of a self contained resonance.

Differentiating the eigenfunction with respect to time gives

$$\begin{aligned} \frac{\partial}{\partial t} I_n(x,t) &= \Re\left(\gamma_n I_n(x,t)\right) \\ \frac{\partial^2}{\partial t^2} I_n(x,t) &= \Re\left(\gamma_n^2 I_n(x,t)\right) \end{aligned} \tag{5.5}$$

The Green's function - no longer time dependent, but now frequency specific, becomes

$$g(x,a;\gamma_n) \equiv \gamma_n^2 F(x,a) - \gamma_n K(x,a) - G_l R_s(a) \tag{5.6}$$

so that the spatial part of each eigenfunction will satisfy

$$\int_0^h g(x,a;\gamma_n) \bar{I}_n(a)\, da = \bar{I}_n(x) \tag{5.7}$$

which we can write as

$$\int_0^h \left(g(x,a;\gamma_n) - \delta(x-a)\right) \bar{I}_n(a)\, da = 0 \tag{5.8}$$

As with the more familiar case of a matrix equation, the eigenfunctions occur for values of γ_n which cause the determinant of the above operator to be zero. It can be shown that the so called Fredholm determinant is given by the series

$$\begin{aligned}
&1 - \frac{1}{1!}\int_0^h g(a,a;\gamma_n)\,da \\
&+ \frac{1}{2!}\int_0^h\int_0^h \begin{vmatrix} g(a_1,a_1;\gamma_n) & g(a_1,a_2;\gamma_n) \\ g(a_2,a_1;\gamma_n) & g(a_2,a_2;\gamma_n)\end{vmatrix}\,da_1\,da_2 \\
&- \frac{1}{3!}\int_0^h\int_0^h\int_0^h \begin{vmatrix} g(a_1,a_1;\gamma_n) & g(a_1,a_2;\gamma_n) & g(a_1,a_3;\gamma_n) \\ g(a_2,a_1;\gamma_n) & g(a_2,a_2;\gamma_n) & g(a_2,a_3;\gamma_n) \\ g(a_3,a_1;\gamma_n) & g(a_3,a_2;\gamma_n) & g(a_3,a_3;\gamma_n)\end{vmatrix}\,da_1\,da_2,\,da_3 \\
&+ \ldots
\end{aligned} \tag{5.9}$$

The values of $\gamma_n = j\omega_n - \alpha_n$ for which the above series sums to zero provide the angular frequencies ω_n and the decay rates $\alpha_n = \omega_n/(2Q_n)$ for each of the infinite number of free resonances. Figure 5.2 shows the landscape of $det(\hat{A} - \hat{1})$ by indicating which quadrant of the complex plane contains the determinant. The normal modes occur at the points where all four quadrants meet - there are three in the figure.

In practice, the functions $F(x,y)$, $K(x,y)$, and $R_s(x)$ tend to become available in matrix form, employing a finite number of dimensions, the number of which is set by the spatial resolution at which the underlying reactance distributions are available, typically from a few tens up to a few hundred. With the operator in equation 5.8 in the form of a matrix with N dimensions, equating the determinant to zero leads to a polynomial of degree $2N$ in γ_n, the solutions of which determine the first $2N$ eigenfunctions, which emerge as N conjugate pairs. We can see this by writing 5.6 as

$$g(x,a;\gamma_n) \equiv (\alpha^2 - \omega^2)F(x,a) + \alpha K(x,a) - G_l R_s(a) - j\omega\left(2\alpha F(x,a) + K(x,a)\right) \tag{5.10}$$

Since each imaginary term in the determinant also includes an ω, the determinant will have an odd power of ω as a factor in its imaginary part, and all other terms involving ω appear as even powers. Therefore the zeroes of the determinant are invariant to a change of sign of ω and thus, given any solution to the characteristic equation, its complex conjugate will also be a solution.

6. Resonant Primary Coupling

We introduced in section 2 the coupling operator appropriate for a coupled primary feed. In most Tesla coil situations, the primary inductor is itself resonated with a primary tank capacitance C_p. In this section we will look at how this lumped primary resonator is coupled to the distributed secondary resonator.

For the resonator with a forced primary current $I_{pri}(t)$ we have the distributed secondary current

$$I = \hat{v}I_{pri} + \hat{A}I \tag{6.1}$$

where $\hat{v}$ is the time domain operator $\hat{v_p}$ given in equation 2.16. The induced primary voltage which results from this secondary current is

$$V_{pri}(t) = \int_0^h M_p(a)\frac{\partial}{\partial t}I(a,t)\,da + L_p\frac{d}{dt}I_{pri}(t) + R_p I_{pri}(t) \tag{6.2}$$

where R_p is the primary circuit resistance and L_p is the self inductance of the primary, which we can take as lumped so long as the self resonant frequency of

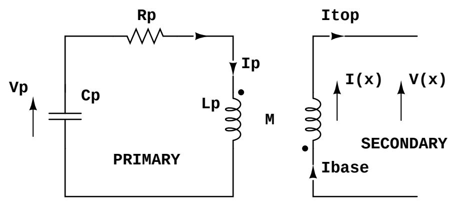

FIGURE 6.1. Primary current convention and winding sense

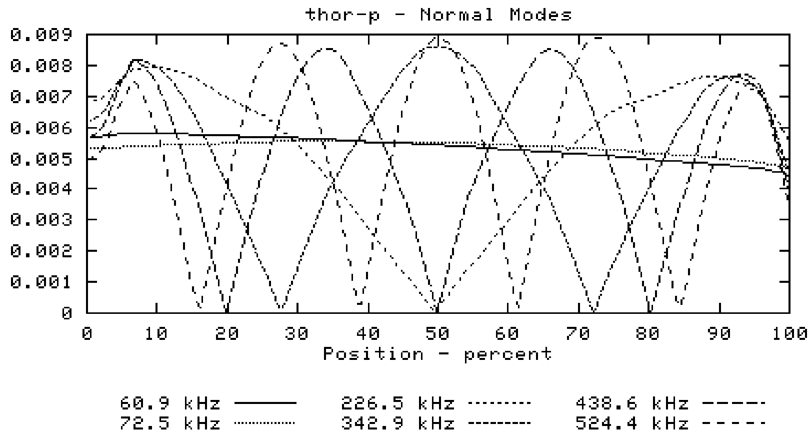

FIGURE 6.2. The first few normal modes of a primary-coupled resonator

the primary inductor is much higher than the frequencies we want to consider. We have assumed the positive sign for the mutual inductance, which is the case when the primary current has the direction, with respect to the winding sense, shown in figure 6.1. If either the primary current or winding sense is reversed, the terms involving $M_p(a)$ in equations 2.2 and 6.2 take on the opposite sign. If the primary inductor is resonated with a primary tank capacitance C_p, then we also have the relationship

$$I_{pri}(t) = -C_p \frac{d}{dt} V_{pri}(t) \tag{6.3}$$

and substituting $V_{pri}(t)$ from equation 6.2 into this gives

$$\begin{aligned} I_{pri}(t) &= -C_p \int_0^h M_p(a) \frac{\partial^2}{\partial t^2} I(a,t)\, da - C_p L_p \frac{d^2}{dt^2} I_{pri}(t) - C_p R_p \frac{d}{dt} I_{pri}(t) \\ &= \hat{u} I + \hat{P} I_{pri} \end{aligned} \tag{6.4}$$

where for convenience we have defined the two operators

$$\begin{aligned} \hat{u} f &\equiv -C_p \int_0^h M_p(a) \frac{\partial^2}{\partial t^2} f(a,t)\, da \\ \hat{P} f &\equiv -C_p L_p \frac{d^2}{dt^2} f(t) - C_p R_p \frac{d}{dt} f(t) \end{aligned} \tag{6.5}$$

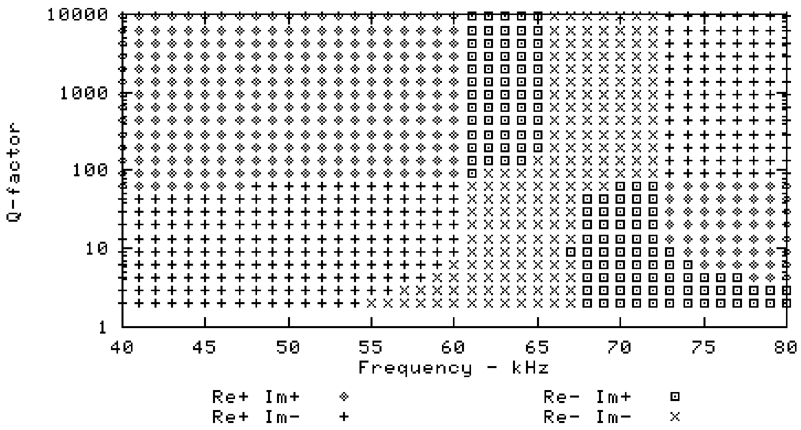

FIGURE 6.3. Landscape of det(AP-1)

We can see that the operators $\hat{P}$ and $\hat{u}$ perform the same role in the primary resonator that $\hat{A}$ and $\hat{v}$ do in the secondary resonator.

For the coupled resonators, this equation and equation 6.1 apply simultaneously, which we can express by the matrix eigenvalue equation

$$\begin{vmatrix} I \\ I_{pri} \end{vmatrix} = \begin{vmatrix} \hat{A} & \hat{v} \\ \hat{u} & \hat{P} \end{vmatrix} \begin{vmatrix} I \\ I_{pri} \end{vmatrix} \tag{6.6}$$

If the integral equation for the isolated secondary is viewed as a coupled system of an infinite number of harmonic resonators $I(x,t)$, then all we have done is to add one more resonator, which brings with it an additional degree of freedom which we accomodate by extending the current vector I by an extra dimension in order to represent the primary oscillator $I_{pri}(t)$.

The resonant modes represented by the eigenfunctions of 6.6 are now the coupled resonances of the primary-secondary system, and will have the separable form

$$\phi_n = \begin{vmatrix} \bar{I} \\ \bar{I}_{pri} \end{vmatrix}_n e^{\gamma_n t} \tag{6.7}$$

where again, γ_n is the complex frequency $j\omega - \alpha$ of the nth mode, so that the actions of the operators $\hat{u}$ and $\hat{P}$ on this particular eigenfunction are

$$\begin{aligned} \hat{u}_n f &\equiv -\gamma_n^2 C_p \int_0^h M_p(a) \bar{f}(a)\, da \\ \hat{P}_n f &\equiv \left(-\gamma_n^2 C_p L_p - \gamma_n C_p R_p\right) \bar{f} \end{aligned} \tag{6.8}$$

Writing 6.6 as

$$\begin{vmatrix} \hat{A} - \hat{1} & \hat{v} \\ \hat{u} & \hat{P} - 1 \end{vmatrix} \begin{vmatrix} I \\ I_{pri} \end{vmatrix} = 0 \tag{6.9}$$

the normal modes occur with singular values of the determinant, ie when γ_n is such that

$$\left(\hat{P} - 1\right)\left(\hat{A} - \hat{1}\right) - \hat{v}_n \hat{u}_n \equiv 0 \tag{6.10}$$

The cross term $\hat{v}_n\hat{u}_n$ represents the primary-secondary coupling, and we can see here that as this reduces to zero, the determinant factorises into the distinct free resonances $\left|\hat{A}-\hat{1}\right| = 0$ and $\left|\hat{P}-1\right| = 0$ of the individual uncoupled resonators. When the coupling term is non-zero, neither of the separate free resonances is a zero of the determinant. It is worthwhile comparing 6.9 with the corresponding equation obtained from two coupled lumped tuned circuits. It is left to the reader to show that in this elementary case,

$$\begin{vmatrix} L_sC_s\gamma^2 + C_sR_s\gamma + 1 & C_sM\gamma^2 \\ C_pM\gamma^2 & L_pC_p\gamma^2 + C_pR_p\gamma + 1 \end{vmatrix} \begin{vmatrix} I_s \\ I_p \end{vmatrix} = 0 \tag{6.11}$$

where the off-diagonal terms will both change sign if the sense of one of the windings is reversed. If we set the individual resonances of the two circuits to the same frequency by a suitable choice of one of the components, then we can make the definitions

$$\omega_0^2 = \frac{1}{L_pC_p} = \frac{1}{L_sC_s}; \quad Q_s = \frac{L_s\omega_0}{R_s}; \quad Q_p = \frac{L_p\omega_0}{R_p}; \quad k^2 = \frac{M^2}{L_pL_s}; \tag{6.12}$$

which reduces the equation of the lumped case to

$$\begin{vmatrix} z^2 + \frac{1}{Q_s}z + 1 & \frac{M}{L_s}z^2 \\ \frac{M}{L_p}z^2 & z^2 + \frac{1}{Q_p}z + 1 \end{vmatrix} \begin{vmatrix} I_s \\ I_p \end{vmatrix} = 0 \tag{6.13}$$

where we have also made a change of coordinates to a normalised complex frequency $z = \gamma/\omega_0$. The characteristic polynomial is then

$$(z^2 + \frac{1}{Q_s}z + 1)(z^2 + \frac{1}{Q_p}z + 1) - k^2z^4 = 0 \tag{6.14}$$

If we compare this equation with 6.10 we can identify a distributed coupling coefficient as an operator $\hat{k}$ which is the square root of the product of the two coupling operators $\hat{v}$ and $\hat{u}$, ie

$$\hat{k}^2 = \frac{\omega^4}{\gamma^4}\hat{v}_n\hat{u}_n \tag{6.15}$$

Techniques for finding the complex frequencies γ_n which satisfy 6.10 will be described later, but given that they can be calculated, we can express any resonant behaviour of the coupled system by a linear sum of the eigenfunctions, as in

$$\begin{vmatrix} I(x,t) \\ I_{pri}(t) \end{vmatrix} = \sum_{n=1}^{\infty} \Re\left\{ \bar{\lambda}_n \begin{vmatrix} \bar{I}(x) \\ \bar{I}_{pri} \end{vmatrix}_n e^{\gamma_n t} \right\} \tag{6.16}$$

with the corresponding voltage distribution given by equations 6.2 and 2.14. The only degrees of freedom left to us are the complex mode amplitudes $\bar{\lambda}_n$ and we must choose these so that at $t = 0$ the above expression delivers a set of instantaneous voltages and currents

$$\{I_{pri}(0), \quad I(x,0), \quad V_{pri}(0), \quad V(x,0)\} \tag{6.17}$$

which match a given set of starting conditions. Note that we need to specify both the currents and the voltages in order to completely describe the instantaneous state of the resonator. A typical set of initial conditions would be

$$\{0, \quad 0, \quad V_{bang}, \quad 0\} \tag{6.18}$$

corresponding to an idle resonator armed with an initial primary charge $C_p V_{bang}$. The process by which 6.17 is decomposed into a set of mode amplitudes $\{\gamma_n\}$, which we shall look at in the next section, is the key to time domain modeling of the resonator, because once this step is completed, the subsequent time evolution of the resonator can be calculated trivially from 6.16.

7. Modal Decomposition

Our task in this section is to show how the mode amplitudes λ_n are chosen to match a set of initial conditions. Note that although we refer to *initial* conditions, we could just as easily set the mode amplitudes to match a final or some intermediate state of the resonator. A general purpose set of initial conditions is set out in 6.17 and we want to find the complex amplitudes λ_n which satisfy the simultaneous equations

$$\begin{aligned} \left|\begin{matrix} I(x,t) \\ I_{pri}(t) \end{matrix}\right|_{t=0} &= \sum_{n=1}^{\infty} \Re\left\{ \bar{\lambda}_n \left|\begin{matrix} \bar{I}(x) \\ \bar{I}_{pri} \end{matrix}\right|_n \right\} \\ \left|\begin{matrix} V(x,t) \\ V_{pri}(t) \end{matrix}\right|_{t=0} &= \sum_{n=1}^{\infty} \Re\left\{ \bar{\lambda}_n \left|\begin{matrix} \bar{V}(x) \\ \bar{V}_{pri} \end{matrix}\right|_n \right\} \end{aligned} \tag{7.1}$$

In practice, $I(x,t)$, along with $\bar{I}_n(x)$, are available as column vectors with N components resulting from a numerical determination of the solenoid operator to a spatial resolution of N elements. The left hand side of 7.1 can be taken as a single column vector of 2N+2 real numbers which describe the starting conditions. The column vectors on the right hand side have the same number of dimensions. Typically N may be around 200. We also usually want to restrict the number of eigenfunctions to the first m, perhaps 10 or 20, modes. Since $m < N+1$, the system 7.1 is underdetermined, meaning there are too few eigenfunctions available to exactly satisfy all $2N+2$ equations. Therefore we must apply some technique such as least squares fitting in order to choose a set of $\{\lambda_n\}$ which provide a best fit to the starting conditions using the limited number m of available eigenfunctions. We'll use the following abbreviations

$$\begin{aligned} \left|\begin{matrix} I(x,t) \\ I_{pri}(t) \\ V(x,t) \\ V_{pri}(t) \end{matrix}\right|_{t=0} &= I(k); \quad k = 1 \ldots 2N+2 \\ \left|\begin{matrix} \bar{V}(x) \\ \bar{V}_{pri} \\ \bar{V}(x) \\ \bar{V}_{pri} \end{matrix}\right|_n &= \phi_n(k) \end{aligned} \tag{7.2}$$

where we now use $I(k)$ to refer to an element of the entire set of initial conditions, rather than just a current. Our choice of a set $\{\lambda_n\}$ results in an estimate of the initial conditions $I(k)$ given by

$$\Psi(k) = \sum_{n=1}^{m} \Re\{\lambda_n \phi_n(k)\} \tag{7.3}$$

and the least squares error resulting from this estimate is

$$\chi^2 = \sum_{k=1}^{2N+2} \left(I(k) - \Psi(k)\right)^2 W(k) \tag{7.4}$$

The real weighting coefficients $W(k)$ will be discussed later - for now the reader can take them to be unity. Ideally we would like to find a set $\{\lambda_n\}$ which gives a zero χ^2, but we have only m eigenfunctions, rather than the $N+1$ which would be required for this. Therefore we must settle for minimising χ^2. This we can do by choosing $\{\lambda_n\}$ so that χ^2 is at a turning point with respect to variations of each of the λ_n. By setting the m partial differentials of χ^2 to zero, we get the set of m real equations

$$\delta(\chi^2) = \sum_{k=1}^{2N+2} 2\left(\Psi(k) - I(k)\right) W(k) \frac{\partial \Psi(k)}{\partial \lambda_j} \delta\lambda_j = 0; \quad j = 1 \ldots m \tag{7.5}$$

for any complex variation $\delta\lambda_j$ of the jth mode amplitude. Noting that

$$\frac{\partial \Psi(k)}{\partial \lambda_j} \delta\lambda_j = \Re\{\phi_j(k)\delta\lambda_j\} \tag{7.6}$$

and remembering that $\Psi(k)$ and $I(k)$ are both real, we can write 7.5 as

(7.7)

$$\Re\left\{\delta\lambda_j \sum_{k=1}^{2N+2} \Psi(k)\phi_j(k)W(k) - \delta\lambda_j \sum_{k=1}^{2N+2} I(k)\phi_j(k)W(k)\right\} = 0; \quad j = 1 \ldots m$$

The second term on the left is just the weighted inner product of I and ϕ_j, which we'll abbreviate as

$$C_j = \sum_{k=1}^{2N+2} I(k)\phi_j(k)W(k) \tag{7.8}$$

C_j is a complex scalar which measures the component of the initial conditions vector I which lies in the direction of the jth eigenvector, or equivalently, the similarity or correlation between the shape of the initial conditions distribution and that of the mode j. Using this, our least squares minimisation condition becomes

$$\Re\left\{\delta\lambda_j \sum_{k=1}^{2N+2} \Psi(k)\phi_j(k)W(k) - \delta\lambda_j C_j\right\} = 0; \quad j = 1 \ldots m \tag{7.9}$$

and for this to be true for any arbitrary variation of the λ_j, we must satisfy the m complex conditions

$$\sum_{k=1}^{2N+2} \Psi(k)\phi_j(k)W(k) - C_j = 0; \quad j = 1 \ldots m \tag{7.10}$$

Expanding $\Psi(k)$ by 7.3, we get the necessary conditions on the λ_j themselves,

$$\sum_{k=1}^{2N+2} \sum_{n=1}^{m} \Re\{\lambda_n \phi_n(k)\}\phi_j(k)W(k) - C_j = 0; \quad j = 1 \ldots m \tag{7.11}$$

If we take λ_j to have the real and imaginary components defined by

$$\lambda_j = a_j + jb_j \tag{7.12}$$

we have

$$\sum_{k=1}^{2N+2}\sum_{n=1}^{m}(a_n\Re\{\phi_n(k)\} - b_n\Im\{\phi_n(k)\})\phi_j(k)W(k) - C_j = 0; \quad j = 1\ldots m \tag{7.13}$$

If we define two matrices

$$B'_{jn} = \sum_{k=1}^{2N+2}\Re\{\phi_n(k)\}\phi_j(k)W(k) \quad \text{and} \quad B''_{jn} = \sum_{k=1}^{2N+2}\Im\{\phi_n(k)\}\phi_j(k)W(k) \tag{7.14}$$

we can write 7.13 as

$$\sum_{n=1}^{m}\left(a_n B'_{jn} - b_n B''_{jn}\right) - C_j = 0; \quad j = 1\ldots m \tag{7.15}$$

If we take the real and imaginary parts of 7.15 separately, we obtain a set of $2m$ real simultaneous equations in the $2m$ variables a_n and b_n, which we can represent by the matrix equation

(7.16)

$$\begin{vmatrix} \Re\{B'_{11}\} & \ldots & \Re\{B'_{1m}\} & -\Re\{B''_{11}\} & \ldots & -\Re\{B''_{1m}\} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Re\{B'_{mm}\} & \ldots & \Re\{B'_{mm}\} & -\Re\{B''_{m1}\} & \ldots & -\Re\{B''_{mm}\} \\ \Im\{B'_{11}\} & \ldots & \Im\{B'_{1m}\} & -\Im\{B''_{11}\} & \ldots & -\Im\{B''_{1m}\} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Im\{B'_{m1}\} & \ldots & \Im\{B'_{mm}\} & -\Im\{B''_{m1}\} & \ldots & -\Im\{B''_{mm}\} \end{vmatrix} \begin{vmatrix} a_1 \\ \vdots \\ a_m \\ b_1 \\ \vdots \\ b_m \end{vmatrix} = \begin{vmatrix} \Re\{C_1\} \\ \vdots \\ \Re\{C_m\} \\ \Im\{C_1\} \\ \vdots \\ \Im\{C_m\} \end{vmatrix}$$

which is easily solved by Gaussian elimination to obtain the a_n and b_n, from which the complex mode amplitudes are given by 7.12. What we have done is to obtain the unique set of mode amplitudes which come closest, in the sense of equation 7.4, to the initial conditions at $t = 0$. We have some control over the meaning of the word *closest*, by virtue of the weighting coefficients $W(k)$. If these were absent or set to unity, we would have the situation that a 1 volt error in say, the top voltage, would be considered just as serious by the least squares estimate as a 1 amp error in the say, the base current, thus matching the top voltage to the initial conditions to a much higher precision than the base current. We would rather have a roughly equal precision at all points in the system, and we achieve this by choosing a set of weighting coefficients which are inversely proportional to the square of the average magnitudes of the available m modes. In other words

$$W(k) = \frac{1}{\sum_{j=1}^{m}|\phi_j(k)|}^{2}; \quad k = 1\ldots 2N+2 \tag{7.17}$$

is a suitable weighting to achieve a uniform precision throughout the resonator.

8. Finding Eigenvalues

In section 6, the integral equation describing the coupled primary-secondary system was presented in the form of equation 6.9, reproduced here,

$$\begin{vmatrix} \hat{A} - \hat{1} & \hat{v} \\ \hat{u} & \hat{P} - 1 \end{vmatrix} \begin{vmatrix} I \\ I_{pri} \end{vmatrix} = 0 \tag{8.1}$$

in which $\hat{A}$ is the solenoid operator of the secondary defined by equation 3.1, $\hat{v}$ is the operator describing the coupling of the primary to the secondary defined by equation 2.16, $\hat{P}$ defined in 6.5 describes the primary self coupling and $\hat{u}$ gives the coupling between the secondary and the primary. All of these operators, when expressed in the frequency domain, are quadratic functions of the complex frequency γ, so equation 8.1 represents a quadratic eigenvalue problem. In order to solve for the complex frequency γ, we need to put 8.1 into the form

$$\left(\gamma^2\Theta_2 + \gamma\Theta_1 + \Theta_0\right)\begin{vmatrix}\bar{I}\\ \bar{I}_{pri}\end{vmatrix} = 0 \tag{8.2}$$

where the Θ are matrices of operators acting on the column vector of complex amplitudes $\begin{vmatrix}\bar{I}\\ \bar{I}_{pri}\end{vmatrix}$. The Θ matrices are given by

$$\Theta_2 = \begin{vmatrix}\hat{C}\hat{L} & -\hat{C}\int_0^r M_p(a)\,da\\ -C_p\int_0^h M_p(a)\bar{f}(a)\,da & -C_pL_p\end{vmatrix} \tag{8.3}$$

$$\Theta_1 = \begin{vmatrix}\hat{C}\hat{R} + \hat{G}_l\hat{L} & -hatG_l\int_0^r M_p(a)\,da\\ 0 & -C_pR_p\end{vmatrix} \tag{8.4}$$

$$\Theta_0 = \begin{vmatrix}-\hat{1} & 0\\ 0 & -1\end{vmatrix} \tag{8.5}$$

Notes and References

[1] The TSSP web site at `http://www.abelian.demon.co.uk/tssp/`

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